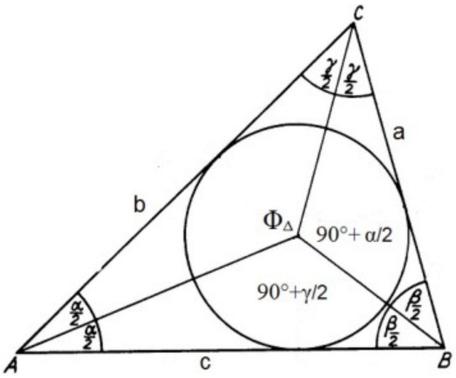
Morley's Trisector Theorem by Gerhard Schallenkamp (23.01.2017)

– school-level of 8th class required –

Notations. $\angle BAC$ denotes the angle ($\leq 180°$) at the vertex A in the triangle ABC.
Φ_Δ denotes the intersection of the three angle bisectors of the triangle Δ.

For preparation some properties of Φ_Δ:

> Theorem. The three angle bisectors intersect in a single point, the incenter, denoted by Φ_Δ, the center of the triangle's incircle, with the angles $90°+\alpha/2$, $90°+\beta/2$, $90°+\gamma/2$. Each of these angles is open to that side of triangle which is opposite to the angles α, β, γ.



Proof: Any point of an angle bisector is equidistant to the sides of the angle. Thus $\Phi_\Delta = \Phi_{ABC}$ is equidistant to sides a, b und c. The angle at Φ_{ABC} opposite side a is $180° - \beta/2 - \gamma/2 = 180° - ½(\beta+\gamma) = 180° - ½(180° - \alpha) = 90° + \alpha/2$.

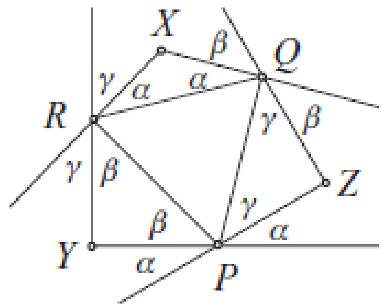
This variation of the theorem will be important:

> Theorem. If the point P lies in the triangle ABC on the angle bisector of A and $\angle BPC = 90° + \alpha/2$, then P is unique and $= \Phi_{ABC}$.

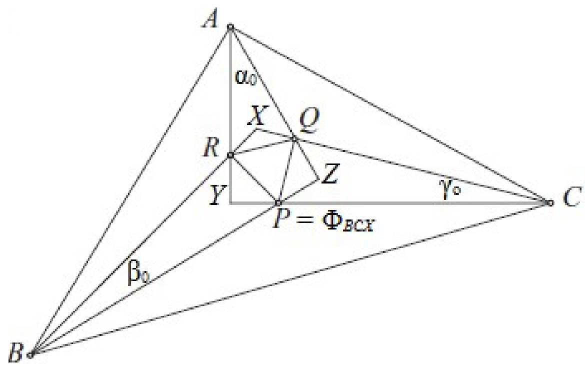
Now we are ready for Frank Morley's theorem:

> Theorem. The three intersection points of **adjacent angle trisectors** in any triangle are the vertices of an **equilateral** triangle. (see outlines below).

Proof: Starting with an equilateral triangle PQR we shall construct a triangle ABC with the arbitrary angles $3\alpha_0$, $3\beta_0$ und $3\gamma_0$. $\alpha_0+\beta_0+\gamma_0 = 60°$ follows from the angle sum of a triangle. We need the angles $\alpha = 60° - \alpha_0$, $\beta = 60° - \beta_0$ und $\gamma = 60° - \gamma_0$, the important angle sum of which is $\alpha+\beta+\gamma = 180° - (\alpha_0+\beta_0+\gamma_0) = 120°$.



The angles α, β, γ are built to the triangle PQR such as in the right outline. By reason of $\alpha+\beta+\gamma+60° = 180°$ straight lines arise in the vertices P, Q und R, which meet in the points X, Y and Z and in the points A, B and C.



Both outlines show: The mid angle at A is $\angle RAQ = 180° - \alpha - (\alpha+\beta+\gamma) = 60° - \alpha = \alpha_0$. nalogously $\angle PBR = \beta_0$ and $\angle PCQ = \gamma_0$.

ξ denotes the angle $\angle BXC$ at X. We calculate $\xi = 180° - 2\alpha = 2\cdot(90° - \alpha)$.

The big angle at P is $180° - \alpha = 90° + (90° - \alpha) = 90° + \xi/2$. Because of symmetry P lies on the angle bisection through X, too. That is why $P = \Phi_{BCX}$.

Therefore the lines BP und CP are angle bisections in the triangle BCX, and therefore $\angle CBP = \angle RBP = \beta_0$ and $\angle BCP = \angle QCP = \gamma_0$. $Q = \Phi_{ACY}$ und $R = \Phi_{ABZ}$, follow analogously, so that we get three equal angles at the point A and analogously at the points B and C. Thus the triangle ABC has the wanted angles and shows the correctness of Morley's theorem.

(Concept and parts of the outlines from the book Claudi Alsina, Roger B. Nelsen, *Charming Proofs*, 2010, p. 100)