

## Morley's Trisector Theorem by Gerhard Schallenkamp (23.01.2017)

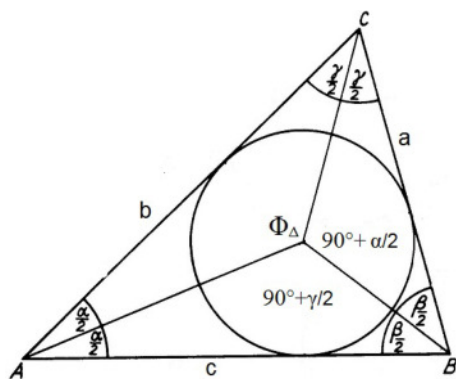
– school-level of 8<sup>th</sup> class required –

Notations.  $\angle BAC$  denotes the angle ( $\leq 180^\circ$ ) at the vertex  $A$  in the triangle  $ABC$ .

$\Phi_\Delta$  denotes the intersection of the three angle bisectors of the triangle  $\Delta$ .

For preparation some properties of  $\Phi_\Delta$ :

Theorem. The three angle bisectors intersect in a single point, the incenter, denoted by  $\Phi_\Delta$ , the center of the triangle's incircle, with the angles  $90^\circ + \alpha/2$ ,  $90^\circ + \beta/2$ ,  $90^\circ + \gamma/2$ . Each of these angles is open to that side of triangle which is opposite to the angles  $\alpha$ ,  $\beta$ ,  $\gamma$ .



Proof: Any point of an angle bisector is equidistant to the sides of the angle. Thus  $\Phi_\Delta = \Phi_{ABC}$  is equidistant to sides  $a$ ,  $b$  and  $c$ . The angle at  $\Phi_{ABC}$  opposite side  $a$  is  $180^\circ - \beta/2 - \gamma/2 = 180^\circ - 1/2(\beta + \gamma) = 180^\circ - 1/2(180^\circ - \alpha) = 90^\circ + \alpha/2$ .

This variation of the theorem will be important:

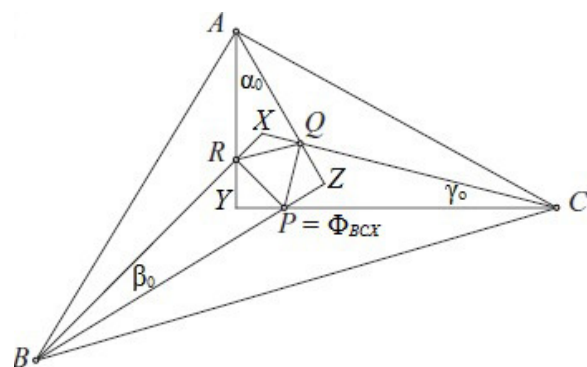
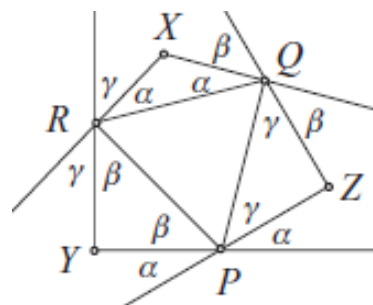
Theorem. If the point  $P$  lies in the triangle  $ABC$  on the angle bisector of  $A$  and  $\angle BPC = 90^\circ + \alpha/2$ , then  $P$  is unique and  $= \Phi_{ABC}$ .

Now we are ready for Frank Morley's theorem:

Theorem. The three intersection points of **adjacent angle trisectors** in any triangle are the vertices of an **equilateral triangle**. (see outlines below).

Proof: Starting with an equilateral triangle  $PQR$  we shall construct a triangle  $ABC$  with the arbitrary angles  $3\alpha_0$ ,  $3\beta_0$  and  $3\gamma_0$ .  $\alpha_0 + \beta_0 + \gamma_0 = 60^\circ$  follows from the angle sum of a triangle. We need the angles  $\alpha = 60^\circ - \alpha_0$ ,  $\beta = 60^\circ - \beta_0$  and  $\gamma = 60^\circ - \gamma_0$ , the important angle sum of which is  $\alpha + \beta + \gamma = 180^\circ - (\alpha_0 + \beta_0 + \gamma_0) = 120^\circ$ .

The angles  $\alpha$ ,  $\beta$ ,  $\gamma$  are built to the triangle  $PQR$  such as in the right outline. By reason of  $\alpha + \beta + \gamma + 60^\circ = 180^\circ$  straight lines arise in the vertices  $P$ ,  $Q$  and  $R$ , which meet in the points  $X$ ,  $Y$  and  $Z$  and in the points  $A$ ,  $B$  and  $C$ .



Both outlines show: The mid angle at  $A$  is  $\angle RAQ = 180^\circ - \alpha - (\alpha + \beta + \gamma) = 60^\circ - \alpha = \alpha_0$ . Analogously  $\angle PBR = \beta_0$  and  $\angle PCQ = \gamma_0$ .

$\xi$  denotes the angle  $\angle BXC$  at  $X$ . We calculate  $\xi = 180^\circ - 2\alpha = 2 \cdot (90^\circ - \alpha)$ .

The big angle at  $P$  is  $180^\circ - \alpha = 90^\circ + (90^\circ - \alpha) = 90^\circ + \xi/2$ . Because of symmetry  $P$  lies on the angle bisection through  $X$ , too. That is why  $P = \Phi_{BCX}$ .

Therefore the lines  $BP$  and  $CP$  are angle bisections in the triangle  $BCX$ , and therefore  $\angle CBP = \angle RBP = \beta_0$  and  $\angle BCP = \angle QCP = \gamma_0$ .  $Q = \Phi_{ACY}$  and  $R = \Phi_{ABZ}$ , follow analogously, so that we get three equal angles at the point  $A$  and analogously at the points  $B$  and  $C$ . Thus the triangle  $ABC$  has the wanted angles and shows the correctness of Morley's theorem.

(Concept and parts of the outlines from the book Claudi Alsina, Roger B. Nelsen, *Charming Proofs*, 2010, p. 100)